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A BIFURCATION PHENOMENON IN AN ELASTIC–PLASTIC SYMMETRICAL SHALLOW TRUSS SUBJECTED TO A SYMMETRICAL LOAD

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Abstract—This paper analyses the large deformation of an elastic–plastic symmetrical shallow truss subjected to a static symmetrical load. An energy criterion is provided to examine the stability of the symmetrical deformation path of the truss. For a truss made of elastic–perfectly plastic material, the stable non-symmetrical deformation path and load–deflection curve are determined.

NOTATION

A	cross-sectional area of each bar
a	half span of the truss
d	deviation of hinge A from the symmetrical axis
E	Young's modulus
E_t	tangent modulus
H, h	height of the truss
K	parameter defined by eqn (5)
l	length of the bar
P	vertical load applying to hinge A
r	polar radius, the length of bar AB
ΔU	variation in strain energy of the truss
Y	yield stress
x, y	Cartesian coordinates of hinge A
ε	strain
θ	polar angle, the angle between bar AB and the x-axis
ρ	length of bar AC
σ_1, σ_2	compressive stresses of bars AB and AC, respectively
φ	angle between bar AC and the negative x-axis.

1. INTRODUCTION

The uniqueness and stability of the solutions in elastic–plastic problems have received a great deal of attention over the years. It is well known that the solution to a well-defined boundary-value problem in elasticity is unique. Hill[1] showed that the elastic–plastic boundary-value problem always has a unique solution when the total strain remains small, the positional changes and rotations of material elements are neglected, the work-hardening is monotonic and the yield function is identical to potential. But in the case of large deformation, a perfect uniqueness criterion has not been found so far. Hill[2] proposed a sufficient condition on uniqueness and an extremum principle characterizing the uniqueness. He found that the solution is unique only when the rate of hardening exceeds a certain critical value which depends on the particular solution. Hill[3(a), 3(b)] also investigated problems of rigid–plastic solids and obtained some similar results. White and Hodge[4, 5] studied the structures of elastic–perfectly plastic material and found that non-unique solutions may exist for some problems with small deformation and zero strain hardening even for loads below the yield point load. They thought that this non-uniqueness phenomenon stems directly from the elastic–perfectly plastic material idealization, as the stress–strain relationship is not unique in this model when stress exceeds the yield point. Thus they concluded that if the stress–strain relationship is unique, then the solution for the structure will be unique. But recent research has shown that in non-linear problems, the solution may be nonunique even though all control equations and relations are uniquely prescribed and this is indeed due to the non-linear property of the structure. This paper discusses the

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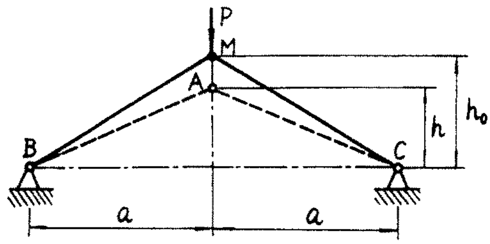

Fig. 1. The symmetrical elastic–perfectly plastic shallow truss investigated by Huang and Tsai[6].

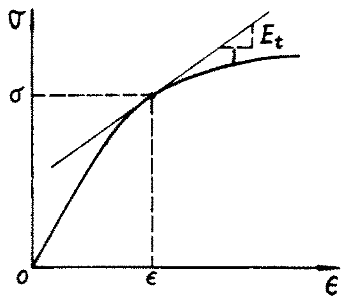

Fig. 2. Stress–strain relationship of a general elastic–plastic material with isotropic properties in tension and compression.

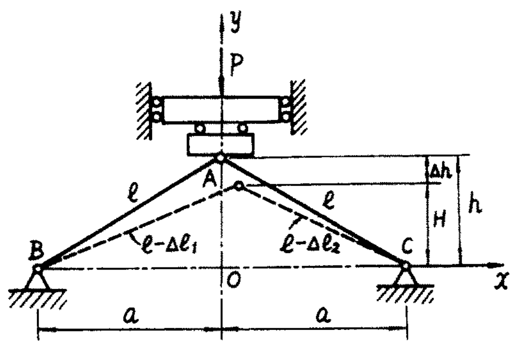

Fig. 3. The loading mechanism and the coordinate system of an elastic–plastic shallow truss.

stability of a symmetrical deformation for a symmetrical shallow truss subjected to a symmetrical load and provides an example of equilibrium bifurcation caused by geometrical and material nonlinearities.

2. BIFURCATION CRITERION FOR THE DEFORMATION PATH OF A SHALLOW TRUSS

A symmetrical pin-jointed weightless shallow truss with span $2a$, initial height h and with a mass M attached at the middle hinge A is shown in Fig. 1. Assuming that the truss is made of elastic–perfectly plastic material, Huang and Tsai[6] studied its dynamic snap-through process when an impulsive and a step load were applied at the middle hinge. In their work, only the symmetrical path is considered, in which, hinge A is supposed to move along the symmetrical axis of the structure throughout the deformation process. However, our analysis will show that in the case of static loading, the symmetrical path is an unstable equilibrium path.

A general elastic–plastic material with isotropic properties in tension and compression is considered in the following analysis. Its stress–strain relationship is shown in Fig. 2, where E_t is the tangent modulus. The loading mechanism and the coordinate system of the truss are shown in Fig. 3. It is assumed that hinge A is moved downwards quasi-statically and $y_A < 0$. There is no restraint and no friction for its motion in the x-axis.

When hinge A is located on the y-axis at $y_A = h$, the truss is in a symmetrical state. In the next stage when y reduces to $H = h - \Delta h$ with $\Delta h \ll h$, suppose that both bars are not

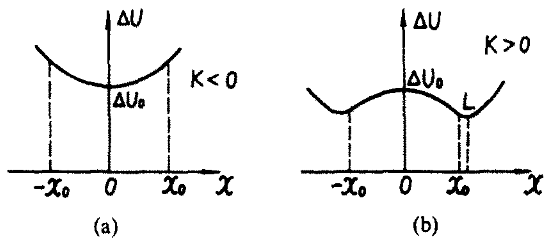

Fig. 4. Relationship between the change of strain energy and the horizontal deviation of hinge A during the variation of its height from $y_A = h$ to $h-\Delta h$ for: (a) $K < 0$; (b) $K > 0$.

unloaded but deformation may be nonsymmetrical with bar AB and AC shorter by Δl_1 and Δl_2, respectively, the variation in strain energy of the truss is

$$\Delta U = A\sigma(\Delta l_1 + \Delta l_2) + \frac{A}{2l} E_t(\Delta l_1^2 + \Delta l_2^2) \tag{1}$$

where A is the cross-sectional area of each bar, σ and l are the compressive stress and the length of the bars when $y_A = h$, and E_t is the tangent modulus of material when the stress is σ.

From the geometric relationship shown in Fig. 3, we have

$$2a = (l^2 - H^2)^{1/2}\left\{2 - \frac{l}{l^2 - H^2}(\Delta l_1 + \Delta l_2) - \frac{1}{2}\frac{H^2}{(l^2 - H^2)^2}(\Delta l_1^2 + \Delta l_2^2) + O(\Delta l_1^3) + O(\Delta l_2^3)\right\} \tag{2}$$

so that

$$\frac{1}{2l}(\Delta l_1^2 + \Delta l_2^2) = C - \frac{l^2 - H^2}{H^2}(\Delta l_1 + \Delta l_2) + O(\Delta l_1^3) + O(\Delta l_2^3) \tag{3}$$

where C is a constant depending on l and H but independent of Δl_1 and Δl_2. Substituting eqn (3) into eqn (1) results in

$$\Delta U = A\left(\sigma - E_t \frac{l^2 - H^2}{H^2}\right)(\Delta l_1 + \Delta l_2) + C' + O(\Delta l_1^3) + O(\Delta l_2^3) \tag{4}$$

where C' is independent of Δl_1 and Δl_2.

Geometry indicates that for the same value of Δh, the value of $\Delta l_1 + \Delta l_2$ is maximum when hinge A is on the y-axis. Therefore, if

$$K = \sigma - E_t \frac{l^2 - H^2}{H^2} < 0$$

and if the deformation is symmetrical, the value of ΔU defined by eqn (4) is a minimum (see Fig. 4(a)). Accordingly, the required load $P = \Delta U/\Delta h$ is a minimum, then the symmetrical path is a stable equilibrium path. On the contrary, if $K > 0$, the symmetrical path will be unstable (see Fig. 4(b)).

Since $\Delta h \ll h$, we have

$$K = \sigma - E_t \frac{l^2 - H^2}{H^2} = \sigma - E_t \frac{a^2}{h^2}. \tag{5}$$

At the beginning of deformation, $\sigma = 0$ and $E_t > 0$, so that $K < 0$ always holds. Therefore, the truss deforms in a symmetrical path at the first phase and the symmetrical

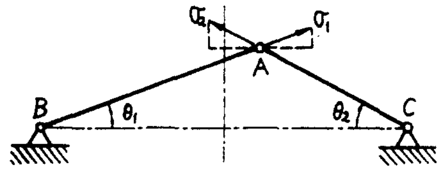

Fig. 5. Equilibrium of hinge A when it deviates from the symmetrical axis.

deformation is stable. Whether the symmetrical path becomes unstable or not depends on whether K changes sign or not before $y_A = 0$.

For elastic–perfectly plastic material, when the stresses on both bars reach the yield point, it is obvious that $E_t = 0$, and $K > 0$. This implies that the truss will become unstable. For strain-hardening material, when the hardening modulus E is sufficiently small, the truss may also become unstable due to $K > 0$. In summary, $K < 0$ represents a sufficient and necessary condition for the stability of any symmetrical deformation state of the truss. This result is in agreement with Hill's idea[2] on the sufficient condition for stability of more general cases.

3. NON-SYMMETRICAL DEFORMATION PATH AFTER BIFURCATION

In the following, the stable non-symmetrical path taken by hinge A after bifurcation for an elastic–perfectly plastic truss will be determined.

It can be proved that one of the bars must be unloaded when hinge A departs from the symmetrical axis. A straightforward physical explanation for elastic–perfectly plastic material is as follows. In Fig. 5, if hinge A departs from the symmetrical axis but none of the bars is unloaded, then after bifurcation, the compressive stresses in the two bars, σ_1 and σ_2, will equal the yield stress, i.e. $\sigma_1 = \sigma_2 = Y$. Since $\theta_1 < \theta_2$, we have $\sigma_1 \cos\theta_1 > \sigma_2 \cos\theta_2$, which means that the resultant horizontal force applied to hinge A is in the direction deviating from the symmetrical axis. Therefore, if there exists a stable equilibrium state after hinge A departs from the symmetrical path, one of the bars must unload. This can also be shown by examining the change of strain energy. In Fig. 3, the bars yield when hinge A reaches $y_A = h$ and the length of each bar is l. Suppose hinge A has a horizontal deviation of x during a small variation of its height from $y_A = h$ to $h - \Delta h$ ($\Delta h \ll h$) after bifurcation, the strain energy of the truss is: for $-x_0 < x < x_0$

$$\Delta U = AY\left(\frac{2h}{l}\Delta h - \frac{a^2}{l^3}\Delta h^2 - \frac{h^2}{l^3}x^2\right) \tag{6a}$$

and for $x < -x_0$ or $x > x_0$

$$\begin{aligned}\Delta U = {} & \frac{EA}{2l^3}\left(1 - \frac{Y}{E}\right)^2 (ax - h\Delta h)^2 \\ & - \frac{AY}{l}\left(1 - \frac{Y}{E}\right)\left[(ax - h\Delta h) + (x^2 + \Delta h^2) + \frac{1}{l^2}(ax - h\Delta h)^2\right] \\ & + \frac{AY^2}{2lE}\left[(ax - h\Delta h) + \frac{1}{2l^2}(hx + a\Delta h)^2\right] \\ & + \frac{AY}{l}\left[(ax + h\Delta h) - \frac{1}{2l^2}(hx - a\Delta h)^2\right]\end{aligned} \tag{6b}$$

where

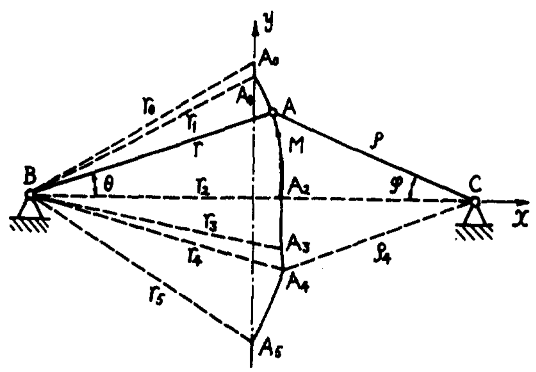

Fig. 6. Non-symmetrical deformation path of the truss after bifurcation.

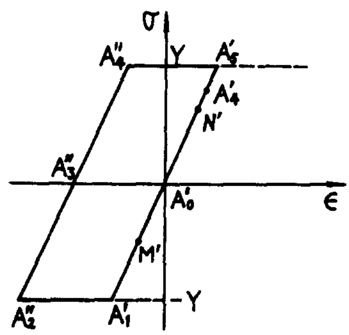

Fig. 7. Stress–strain states of the bars in the process of deformation.

$$x_0 = \frac{h}{a}\Delta h - \frac{l^2}{2a^3}\Delta h^2 > 0.$$

For $-x_0 < x < x_0$, none of the bars are unloaded. But for $x < -x_0$ or $x > x_0$, one of the bars is unloaded. It can be seen that U has a minimum for $x < -x_0$ or $x > x_0$ (see point L in Fig. 4(b)), which corresponds to a stable equilibrium state. Hence, for the elastic–perfectly plastic truss, if none of the bars are unloaded, no static equilibrium of non-symmetrical deformation path exists. Therefore, we need to investigate the case of unloading in one bar after bifurcation.

Suppose that bar AB is unloaded (see Fig. 5), for the non-symmetrical deformation path, the equilibrium for hinge A in the horizontal direction leads to

$$\sigma_1 \cos\theta_1 = \sigma_2 \cos\theta_2 \tag{7}$$

where σ_1 and σ_2 can be expressed by strains of the bars by means of the constitutive relation of the material. Taking a polar coordinate system with point B being the pole, the polar equation of the deformation path can be deduced from eqn (7). The entire deformation process of the truss can be divided into six phases as indicated in Fig. 6.

Phase I

In this phase, both bars remain elastic and the symmetrical path is stable. As shown in Fig. 7, the states of stresses and strains in the two bars change simultaneously from point A'_0 to A'_1 during this phase.

Phase II

At point A1 in Fig. 6, the stresses in both bars reach the yield stress, i.e. $\sigma_1 = \sigma_2 = Y$. At this moment, E_t changes from the elastic modulus E to zero. From eqn (5), we have $K > 0$, which indicates that the symmetrical path becomes unstable and bifurcation occurs. Let $r_1 = BA_1$, which denotes the polar radius at the bifurcation point, we have

$$E\frac{r_0-r_1}{r_0}=Y$$

i.e.

$$\frac{r_1}{r_0}=1-\frac{Y}{E} \tag{8}$$

where r_0 is the initial length of each bar. Suppose bar AB is unloaded, eqn (7) then leads to the deformation path after bifurcation

$$E\frac{r_0-r}{r_0}\cos\theta=Y\cos\varphi \tag{9}$$

in which, φ is the angle of bar AC to the x-axis, r and θ are the polar coordinates of point A. Since

$$\begin{cases}\cos\varphi=\dfrac{2a-r\cos\theta}{\rho}\\ \rho=(r^2+4a^2-4ar\cos\theta)^{1/2}\end{cases} \tag{10}$$

with ρ being the length of bar AC, the polar equation of the path A_1A_2 after bifurcation is found to be

$$(r_0-r)(r^2+4a^2-4ar\cos\theta)^{1/2}\cos\theta-(r_0-r_1)(2a-r\cos\theta)=0. \tag{11}$$

Phase II ends at $y_A=0$ (refer to point A_2 in Fig. 6). At this moment

$$\begin{cases}r_2=r_1\\ \rho_2=2a-r_2\end{cases}. \tag{12}$$

The deviation of point A_2 from the symmetrical axis is

$$d_2=r_1-a. \tag{13}$$

Equation (11) indicates that in phase II, bar AB is unloaded first and then re-loaded when A moves from point M to point A_2 in Fig. 6. In the stress–strain diagram (Fig. 7), the states of bar AB moves from point A_1' to point M′ (unloading) and then returns to A_1' (re-loading). The state of bar AC moves from point A_1' to point A_2' monotonically.

Phase III

When the truss continues to deform, both bars are unloaded. The equilibrium equation, eqn (7), gives

$$E\frac{r_0-r}{r_0}\cos\theta=\left(Y-E\frac{\rho-\rho_2}{r_0}\right)\cos\varphi. \tag{14}$$

With eqn (10), we obtain

$$(r_0-r)\cos\theta-\left[\frac{r_0+2a-2r_1}{(r^2+4a^2-4ar\cos\theta)^{1/2}}-1\right](2a-r\cos\theta)=0. \tag{15}$$

Phase III ends when the stresses in the two bars become zero (refer to point A_3 in Fig. 6). At this point

$$\begin{cases} r_3 = r_0 \\ \rho_3 = \rho_2+r_0-r_1 \end{cases}. \tag{16}$$

The deviation of point A_3 from the symmetrical axis is

$$d_3 = r_3\cos\theta_3-a. \tag{17}$$

Substituting

$$\cos\theta_3 = \frac{4a^2+r_3^2-\rho_3^2}{4ar_3}$$

into eqn (17) results in

$$d_3 = d_2\left(1+\frac{r_0-r_1}{a}\right) > d_2. \tag{18}$$

It is found that hinge A will continue to deviate from the symmetrical axis after the truss takes the horizontal configuration. During this phase, the stress–strain state of bar AB moves from A_1' to A_0', which are the initial states (Fig. 7). The states of bar AC moves from A_2'' to A_3'' along an unloading path. At point A_3'', bar AC has zero stress but non-zero plastic strain.

Phase IV

When hinge A continues to move downwards, the stresses in both bars become tensile. The equilibrium equation, eqn (14), still holds in this phase. The polar equation of the deformation path of hinge A is the same as eqn (15).

Phase IV ends when the stress in the bar AC reaches the tensile yield point (refer to point A_4 in Fig. 6). At this point, the length of bar AC is

$$\rho_4 = 2r_0+2a-3r_1 \tag{19}$$

and the deviation of hinge A from the symmetrical axis is

$$d_4 = r_4\cos\theta_4-a > d_3. \tag{20}$$

As shown in Fig. 7, the stress–strain states in bars AB and AC change from A_0' to A_4' and from A_3'' to A_4'', respectively.

Phase V

When hinge A moves downwards further, the tensile stress in bar AC remains to be the yield stress Y, but bar AB is subjected to elastic tension. The equilibrium equation in this phase is

$$E\frac{r-r_0}{r_0}\cos\theta = Y\cos\varphi. \tag{21}$$

This phase continues until bar AB yields in tension. At this point (A_5 in Fig. 6), we have

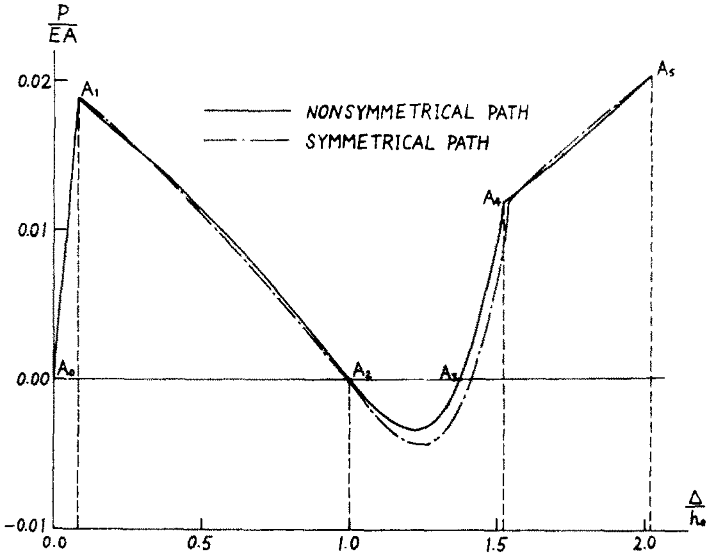

Fig. 8. Relationships between the vertical load P and the vertical displacement Δ of hinge A for $Y/E = 0.02$ and $h_0/r_0 = 0.5$.

$$r_5 = 2r_0 - r_1 \tag{22}$$

$$\cos\theta = \cos\varphi. \tag{23}$$

This implies that hinge A returns to the symmetrical axis. In Fig. 7, the state of bar AC moves to A_5'. In this phase, bar AB is unloaded to point N′ first and then re-loaded to point A_5'. At the end of phase V, the states in the two bars are identical, but they have experienced different deformation history.

Phase VI

After both bars reach the tensile yield stress, if the truss continues to deform, the stresses in both bars will remain constant and the symmetrical deformation path is stable again. It can be seen that for the elastic–perfectly plastic truss, the deviation of hinge A from the symmetrical axis increases first and then decreases after reaching a maximum value of d_4. Finally, hinge A will return to the symmetrical axis.

By the equilibrium condition of hinge A in the y-direction, the relationships between the vertical load P and the vertical displacement of point A, Δ, can be determined and are shown in Fig. 8. The solid line refers to the non-symmetrical deformation path described and the chain line the symmetrical one. Points $A_0, \ldots, A_5$ correspond to the same points in Fig. 6. It is significant that the slope of the P–Δ curve at point A_1 is smaller for the non-symmetrical path than that for the symmetrical one although the load required by the non-symmetrical path is larger than that required by the symmetrical path afterwards. This is a demonstration of the bifurcation at point A_1.

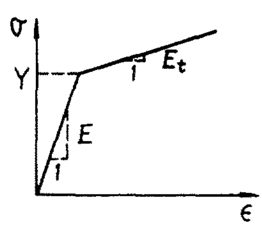

Fig. 9. Constitutive relation of an elastic–linear strain-hardening material.

4. DISCUSSION AND CONCLUDING REMARKS

(1) For a truss made of elastic–linear strain-hardening material, the stress–strain relationship is as shown in Fig. 9. The stability and bifurcation criterion can be given from eqn (5) as

$$\left(1-\frac{E_t}{E}\right)Y+E_t\varepsilon-E_t\frac{a^2}{h^2}>0 \tag{24}$$

where $\varepsilon = (1/2a^2)(h_0^2-h^2)$, and h_0 and h are the heights of the truss before and after deformation, respectively. For the shallow truss ($h \ll a$), eqn (24) leads to

$$E_t < \bar{E}_t \equiv Y\Bigg/\left(\frac{Y}{E}+\frac{a^2}{h^2}-\frac{h_0^2-h^2}{2a^2}\right). \tag{25}$$

Hence, if $E_t < \bar{E}_t$, the symmetrical deformation path of the truss will become unstable and bifurcation will occur.

(2) In setting up the bifurcation criterion, we did not involve any unloading properties of the material. Therefore, the criterion is also valid for non-linear elastic material. For a non-linear elastic material, the constitutive relation is $\sigma = B\varepsilon^n$, in which, B and n are constants, the bifurcation criterion is

$$\varepsilon-n\frac{a^2}{h^2}>0$$

where

$$\varepsilon=\frac{1}{2a^2}(h_0^2-h^2)$$

or

$$n<\bar{n}\equiv\frac{(h_0^2-h^2)(h^2)}{2a^4}. \tag{26}$$

(3) The above analysis is limited to a shallow truss ($h \ll a$). However, the criterion can also be applied to a deep truss at any symmetrical state. Refer to eqn (5), even for linear elastic material ($E_t = E$), when $h \gg a$, bifurcation may occur due to $K > 0$. It should be pointed out that in this case, the stability discussed here is distinct from the stability of a structure in Euler's sense, since the present analysis does not concern bending deformation.

In summary, a perfect criterion on uniqueness and stability has not been proposed for structures in general. This paper has studied an example to examine the existing theories by a new bifurcation phenomenon due to geometrical and material nonlinearities and provided a method to analyse similar problems in more complicated structures.

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